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1992 J. Phys. A: Math. Gen. 25 L799

(http://iopscience.iop.org/0305-4470/25/13/007)

J. Phys. A: Math. Gen. 25 (1992) L799–L806. Printed in the UK

LETTER TO THE EDITOR

On quantum groups for $\mathbb{Z}_N$ models

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Received 21 October 1991, in final form 25 February 1992

Abstract. The quantum group for the $\mathbb{Z}_N$ model is studied from the braid group representation. The fundamental representation is constructed from the Weyl relation $ZX = \omega XZ$ with ω being an Nth root of unity. In the case of $N = 2$ (the eight vertex model), the quantum group is shown to be a homomorphic image of the $\mathrm{GL}_q(2)$ with $q^2 = 1$.

Baxter's eight-vertex model [1] was one of the original sources used in inventing the quantum group. The solutions of the associated Yang–Baxter equation (YBE) are parametrized by elliptic functions [1–4, 13]. They have two degenerate forms: the trigonometric and rational solutions of the YBE. These two degenerate forms give the well known quantum group $\mathrm{SL}_q(2)$ and the Yangian $\mathrm{Y}(\mathrm{SL}(2))$ respectively, which are intensively studied both in mathematics and physics.

Although as early as 1981 Sklyanin [14] carefully defined his quantum algebra of the eight vertex model by analysing a special ansatz for the solution of the YBE, and more recently the Sklyanin algebra is arousing increasing interest, we still do not know whether or not the Sklyanin algebra can be equipped with a Hopf algebra or even a bialgebra structure. One way to get around this problem using a different approach is first to study the quantum group associated with the braid group representation (BGR) or the spectral-parameter independent YBE, since the latter is canonically associated with a bialgebra structure. Then one tries to apply the algebra to the case of the YBE with the spectral-parameter dependency using the so-called Yang–Baxterization procedure.

In the present work we study the quantum group associated with the $\mathbb{Z}_N$ model, which includes the eight vertex model ($N = 2$). The $\mathbb{Z}_N$ model [1, 5, 6, 12] is defined by requiring the Boltzmann weights S^{ij}_{kl}, $i, j, k, l \in \mathbb{Z}_N$ to satisfy the following $\mathbb{Z}_N$ symmetry

$$S^{ij}_{kl} = 0 \qquad \text{unless} \qquad i + j \equiv k + l \pmod N \tag{1}$$

$$S^{i+p,j+p}_{k+p,l+p} = S^{ij}_{kl} \qquad \text{for any} \qquad i, j, k, l, p \in \mathbb{Z}_N. \tag{2}$$

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Moreover the matrix $S = (S^{ij}_{kl}) \in \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$ satisfies the spectral parameter independent YBE or the braid group relation

$$S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23} \tag{3}$$

where $S_{12} = S \otimes 1$, $S_{23} = 1 \otimes S \in \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N)$.

The BGR of the $\mathbb{Z}_N$ model was studied in [7]. We are going to insert an extra parameter $\omega\,(\omega^N = 1)$ into their solutions of the BGR for the $\mathbb{Z}_N$ model. It will be seen that this extra parameter ω plays a similar role to the parameter q in quantum groups.

We will analyse the eight vertex model ($\mathbb{Z}_2$ model) in this context in great detail and show that the quantum group of the $\mathbb{Z}_2$ model is a homomorphic image of the $\mathrm{GL}_q(2)$ with $q^2 = 1$. For general N, we shall give its fundamental representation in terms of the Weyl relations [8]

$$ZX = \omega XZ \qquad Z^N = X^N = 1 \qquad \omega \text{ is an } N\text{th root of unity.} \tag{4}$$

In the case of $N = 2$, we show that the quantum algebra has a Hopf algebra structure by adjoining the determinant. In the general case the algebra is canonically a bialgebra. Whether one can introduce an antipode is not clear at the moment, but we expect a similar situation to $N = 2$ exists.

In the remaining part of the letter we assume the ground field is that of the complex numbers, though all the results are true for any algebraically closed field with characteristic zero.

We start by examining the eight vertex model.

The general S-matrix (or the first-factor transposed R-matrix) for the $\mathbb{Z}_2$ model is of the following type

$$S = \begin{pmatrix} a & & & d \\ & c & b & \\ & b & c & \\ d & & & a \end{pmatrix}. \tag{5}$$

It is easy to check that S satisfies the braid relation (3) if and only if the following conditions hold

$$a^2 = b^2 \qquad c^2 = d^2 \tag{6}$$

From now on we assume that the relation (6) is satisfied. From the methods of quantum inverse scatterings of the Faddeev school [9], the quantum algebra associated with S is an associative algebra $A(S)$ (or sometimes denoted by $A(R)$) generated by $x_{ij}, i, j = 1, 2$ and the unit 1 subject to the following relations

$$S(x \otimes x) = (x \otimes x)S \tag{7}$$

where $(x \otimes x)^{ij}_{kl} = x_{ik} \otimes x_{jl}$.

Proposition 1. If $a \neq c, ad \neq 0$, then the algebra $A(S)$ is an associative algebra generated by $x_{ij}, 1$ with the following relations

$$x_{11}^2 = x_{22}^2 \qquad x_{12}^2 = x_{21}^2 \tag{8}$$
$$x_{11}x_{22} = x_{22}x_{11} \qquad x_{12}x_{21} = x_{21}x_{12} \tag{9}$$
$$x_{11}x_{12} = \omega x_{12}x_{11} \qquad x_{21}x_{22} = \omega x_{22}x_{21} \tag{10}$$
$$x_{11}x_{12} = \omega' x_{21}x_{22} \tag{11}$$
$$x_{11}x_{21} = \omega x_{21}x_{11} \qquad x_{12}x_{22} = \omega x_{22}x_{12} \tag{12}$$
$$x_{11}x_{21} = \omega' x_{12}x_{22} \tag{13}$$

where $\omega = a/b$ and $\omega' = c/d$. In other words, the algebra $A(S)$ is a homomorphic image of the quantum group $\mathrm{GL}_q(2)$ with $q = \omega$. The element $\det = x_{11}x_{22} - \omega x_{12}x_{21}$ *is central.*

Proof. Expanding the matrix equation (7), we obtain the relations (8) and (9) and two other sets of similar relations in $x_{11}, x_{12}, x_{21}, x_{22}$ and $x_{11}, x_{21}, x_{12}, x_{22}$. A typical set of relations in $x_{11}, x_{12}, x_{21}, x_{22}$ takes the following form

$$(a-c)x_{11}x_{12} - bx_{12}x_{11} = -dx_{21}x_{22} \tag{14}$$
$$bx_{11}x_{12} - (a-c)x_{12}x_{11} = dx_{22}x_{21} \tag{15}$$
$$(a-c)x_{21}x_{22} - bx_{22}x_{21} = -dx_{11}x_{12} \tag{16}$$
$$bx_{21}x_{22} - (a-c)x_{22}x_{21} = dx_{12}x_{11}. \tag{17}$$

From substitution of (14) and (15) into (16) it follows that

$$-2a(a-c)x_{11}x_{12} + 2b(a-c)x_{12}x_{11} = 0$$

which is the first relation in (10) provided $a \neq c$. Other relations in (10) and (12) are shown similarly. The relations (11) and (13) then follow by back substitutions into (14)–(17) and the similar equations. □

We add the inverse of det to the algebra $A(S)$, which will then become a Hopf algebra with the antipode γ given by

$$\gamma \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \det{}^{-1} \begin{pmatrix} x_{22} & -\omega^{-1}x_{12} \\ -\omega x_{21} & x_{11} \end{pmatrix}$$

The fundamental two-dimensional representation π of the algebra $A(S)$ can be constructed from the S-matrix, as suggested in [9–11]. Let

$$\pi(x_{ij})_{kl} = S_{jl}^{ki} \qquad i,j,k,l \in \mathbb{Z}_2 \tag{18}$$

then the braid relation assures that this defines an algebra representation of the algebra $A(S)$.

Explicitly π is given by

$$\pi(x_{11}) = \begin{pmatrix} a & \\ & \omega^{-1}a \end{pmatrix} \qquad \pi(x_{12}) = \begin{pmatrix} & \omega'^{-1}c \\ c & \end{pmatrix}$$

$$\pi(x_{21}) = \begin{pmatrix} & c \\ \omega'^{-1}c & \end{pmatrix} \qquad \pi(x_{22}) = \begin{pmatrix} \omega^{-1}a & \\ & a \end{pmatrix}.$$

In other words, the representation is constructed by the Weyl relation and can be written neatly as follows

$$Z = \begin{pmatrix} 1 & \\ & \omega \end{pmatrix} \qquad X = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \qquad ZX = \omega XZ \qquad \omega^2 = 1 \tag{19}$$

$$\pi(x) = \begin{pmatrix} aZ & \omega' cZX \\ cZX & \omega aZ \end{pmatrix}. \tag{20}$$

Proposition 2. Every non-trivial finite-dimensional irreducible representation of $A(S)$ is two-dimensional.

Proof. Let (π, V) be a finite-dimensional representation of $A = A(S)$. If $Ax = xA$ and $Ay = yA$ are two cyclic ideals annihilating each other such that $xy = 0$, then either $\pi(x) = 0$ or $\pi(y) = 0$. Otherwise we have two non-zero A-invariant subspaces of V, thus $V = \pi(x)V = \pi(y)V$, which is a contradiction.

Based on this fact and relations (8), the representation π must factor through either $A(x_{11} - x_{22})$ or $A(x_{11} + x_{22})$. Moreover, π must also factor through $A(x_{12} - \omega' x_{21})$ or $A(x_{12} + \omega' x_{21})$ correspondingly because of relations (11) and (13), i.e. we need only consider irreducible representations of one of the quotient algebras, say $\overline{A} = A/(x_{11} - x_{22}, x_{12} - \omega' x_{21})$. Notice that the centre of the quotient algebra is

$$C(A/(x_{11} - x_{22}, x_{12} - \omega' x_{21})) = \langle \overline{x}_{11}^2, \overline{x}_{12}^2 \rangle$$

where $\overline{x}_{11}, \overline{x}_{12}$ are the representatives of x_{11}, x_{12} in the quotient algebra $\overline{A}$. Hence $\overline{x}_{11}^2$ and $\overline{x}_{12}^2$ are represented by scalar matrices. The only relations in $\overline{A}$ is

$$\overline{x}_{11}\overline{x}_{12} = \omega \overline{x}_{12}\overline{x}_{11}.$$

Therefore there are only two-dimensional non-trivial irreducible representations [8], since $\overline{x}_{11}, \overline{x}_{12}$ are essentially represented by Weyl operators up to rescaling. □

We now go on to look at the $\mathbb{Z}_N$ model. The braid representations of the $\mathbb{Z}_N$ model was studied in [7], where the explicit forms of the S-matrices were given up to $N = 5$. We first summarize their result as follows.

Let

$$\overline{S}^{ij}_{kl} = \sum_{0\leqslant n\leqslant N-1} w_n \delta^{(N)}(i-l-n)\delta^{(N)}(j-k+n) = \sum_{0\leqslant n\leqslant N-1} \overline{S}^{ij}_{kl}(n) \tag{21}$$

where

$$\delta^{(N)}(j) = \begin{cases} 1 & j \equiv 0\,(\mathrm{mod}\ N) \\ 0 & \text{otherwise.} \end{cases}$$

The S-matrix satisfies the YBE (3), which is in the explicit form

$$\sum_{\lambda\mu\nu} S^{ij}_{\mu\nu} S^{\nu k}_{\lambda n} S^{\mu\lambda}_{lm} = \sum_{\lambda\mu\nu} S^{jk}_{\lambda\mu} S^{i\lambda}_{l\mu} S^{\nu\mu}_{mn}. \tag{22}$$

However we are going to introduce a discrete parameter $\omega, \omega^N = 1$ into the S-matrix $\overline{S}$ as follows.

Our new S-matrix S is defined by

$$S^{ij}_{kl} = \sum_{0\leqslant n\leqslant N-1} w_n \delta^{(N)}(i-l-n)\delta^{(N)}(j-k+n)\omega^{i-k} \tag{23}$$

where ω is an Nth root of unity.

The satisfication of YBE of our new S-matrix is verified from the following observation

$$\begin{aligned}
&\sum_{\lambda\mu\nu} \overline{S}^{ij}_{\mu\nu} \overline{S}^{\nu k}_{\lambda n} \overline{S}^{\mu\lambda}_{lm} \omega^{l+m+n-i-j-k} \\
&\qquad = \sum_{\lambda\mu\nu\alpha\beta\gamma} \overline{S}^{ij}_{\mu\nu}(\alpha) \overline{S}^{\nu k}_{\lambda n}(\beta) \overline{S}^{\mu\lambda}_{lm}(\gamma) \\
&\qquad\qquad \times \omega^{(\lambda+\gamma)+(\mu-\gamma)+(\nu-\beta)-(\nu+\alpha)-(\mu-\alpha)-(\lambda-\beta)} \\
&\qquad = \sum_{\lambda\mu\nu} \overline{S}^{ij}_{\mu\nu} \overline{S}^{\nu k}_{\lambda n} \overline{S}^{\mu\lambda}_{lm}.
\end{aligned}$$

If we arrange the indices of S in the following order

$$(ij) = \quad 11, 12, 21, 13, 22, 31, \ldots, N\,N.$$

Then the matrix S, for example when $N = 3$, takes the following form

$$S = \begin{pmatrix}
w_0 & & & & & & \omega^2 w_1 & \omega w_2 & \\
& w_2 & \omega^2 w_0 & & & & & & \omega^2 w_1 \\
& \omega w_0 & w_1 & & & & & & \omega w_1 \\
& & & w_1 & \omega^2 w_2 & \omega w_0 & & & \\
& & & \omega w_2 & w_0 & \omega^2 w_1 & & & \\
& & & \omega^2 w_0 & \omega w_1 & w_2 & & & \\
\omega w_1 & & & & & & w_2 & \omega^2 w_0 & \\
\omega^2 w_2 & & & & & & \omega w_0 & w_1 & \\
& \omega^2 w_1 & \omega w_2 & & & & & & \omega w_0
\end{pmatrix}. \tag{24}$$

We can form the quantum algebra $A(S)$ as above. The algebra $A(S)$ is an associative algebra generated by N^2 generators $x_{ij}, i, j = 1, \ldots, N$ with unity and subject to the following conditions

$$S(x \otimes x) = (x \otimes x)S \tag{25}$$

where $x = (x_{ij})$.

On the free associative algebra $\mathbb{C}\{x_{ij} | 1 \leqslant i, j \leqslant N\}$ there is a bialgebra structure under the coproduct Δ

$$\Delta(x_{ij}) = \sum_{k=1}^{N} x_{ik} \otimes x_{kj} \tag{26}$$

which is extended linearly and multiplicatively, and $1 \to 1 \otimes 1$. The co-unit is given by $\epsilon = x_{ij} \to 0$.

The bialgebra structure on the free algebra induces a bialgebra structure in a natural way provided that the defining relations (25) are preserved under the coproduct. This can be checked as follows.

The defining relations (25) are written explicitly as

$$\sum_{k,l} S_{kl}^{ij}(x_{km}x_{ln}) = \sum_{k,l}(x_{ik}x_{jl})S_{mn}^{kl}.$$

From which it follows that

$$\begin{aligned}
\sum_{k,l} S_{kl}^{ij}\Delta(x_{km})\Delta(x_{ln})
&= \sum_{k,l,\alpha,\beta} S_{kl}^{ij} x_{k\alpha} x_{l\beta} \otimes x_{\alpha m} x_{\beta n} \\
&= \sum_{\alpha,\beta}\Big(\sum_{k,l} S_{kl}^{ij} x_{k\alpha} x_{l\beta}\Big) \otimes x_{\alpha m} x_{\beta n} \\
&= \sum_{\alpha,\beta}\Big(\sum_{k,l} x_{ik} x_{jl} S_{\alpha\beta}^{kl}\Big) \otimes x_{\alpha m} x_{\beta n} \\
&= \sum_{k,l} x_{ik} x_{jl} \otimes \sum_{\alpha,\beta} S_{\alpha\beta}^{kl} x_{\alpha m} x_{\beta n} \\
&= \sum_{k,l} x_{ik} x_{jl} \otimes \sum_{\alpha,\beta} x_{k\alpha} x_{l\beta} S_{mn}^{\alpha\beta} \\
&= \sum_{k,l,\alpha,\beta} x_{ik} x_{jl} \otimes x_{k\alpha} x_{l\beta} S_{mn}^{\alpha\beta} = \sum_{\alpha,\beta} \Delta(x_{i\alpha})\Delta(x_{j\beta}) S_{mn}^{\alpha\beta}.
\end{aligned}$$

Thus $A(S)$ is well-defined and has a bialgebra structure under the coproduct and co-unit.

We can use Weyl relations to give a finite-dimensional representation for the algebra $A(S)$. Consider the following two operators X, Z on the N-dimensional complex space $\mathbb{C}^N$ with following relations [8]

$$ZX = \omega XZ \qquad Z^N = X^N = 1 \tag{27}$$

which is referred to as the Weyl relation. The operators X, Z generate a finite subgroup in $\mathrm{End}(\mathbb{C}^N)$, and they further generate a finite-dimensional algebra, called a Weyl algebra.

Theorem. The assignment $\pi : x_{ij} \rightarrow w_{j-i} Z X^{j-i} \omega^{i-1}$ gives rise to an N-dimensional representation of the quantum algebra $A(S)$.

Proof. Substituting S^{ij}_{jk} and the assignment of x_{ij} into the defining relations of $A(S)$, we want to show the following identity

$$\sum_{\mu,k,l} w_\mu w_{m-k} w_{n-l} \delta^{(N)}(i-l-\mu)\delta^{(N)}(j-k+\mu)\omega^{i-k} Z X^{m-k} \omega^{k-1} Z X^{n-l} \omega^{l-1}$$
$$= \sum_{\mu,k,l} w_\mu w_{k-i} w_{l-j} \delta^{(N)}(k-n-\mu)\delta^{(N)}(l-m+\mu)$$
$$\times \omega^{k-m} Z X^{k-i} \omega^{i-1} Z X^{l-j} \omega^{j-1}.$$

By using the Weyl relation and postmultiplying both sides by $X^{i+j-m-n}$, we then want to show

$$\sum_{\mu,k,l} w_\mu w_{m-k} w_{n-l} \delta^{(N)}(i-l-\mu)\delta^{(N)}(j-k+\mu) Z^2 X^{(j-k)+(i-l)} \omega^{i+l+k-m-2}$$
$$= \sum_{\mu,k,l} w_\mu w_{k-i} w_{l-j} \delta^{(N)}(k-n-\mu)\delta^{(N)}(l-m+\mu)$$
$$\times Z^2 X^{(k-n)+(l-m)} \omega^{2i+j-m-2}.$$

Finally premultiplying both sides by $\omega^{m-2i-j+2} Z^{-2}$ it follows that the above equation is equivalent to the following relation

$$\sum_{\mu,k,l} w_\mu w_{m-k} w_{n-l} \delta^{(N)}(i-l-\mu)\delta^{(N)}(j-k+\mu)$$
$$= \sum_{\mu,k,l} w_\mu w_{k-i} w_{l-j} \delta^{(N)}(k-n-\mu)\delta^{(N)}(l-m+\mu)$$

which is trivial. □

Remark. A similar argument also show that $\pi'(x_{ij}) = w_{j-i} Z X^{j-i} \omega^{-(j-1)}$ gives another representation of the algebra $A(S)$, which can also be obtained from the following relation

$$\pi'(x_{ij})_{kl} = S^{ki}_{jl}.$$

The latter representation is referred as the fundamental one given the reference to the S-matrix. For example, when $N = 3$ we have

$$\pi'(x) = \begin{pmatrix} w_0 Z & w_1 Z X & w_2 Z X^2 \\ w_2 \omega Z X^2 & w_0 \omega Z & w_1 \omega Z X \\ w_1 \omega^2 Z X & w_2 \omega^2 Z X^2 & w_0 \omega^2 Z \end{pmatrix}$$

It is interesting to notice that the Weyl algebra is also a representation of the multiparameter quantum group $\mathrm{GL}_{p_{ij},q_{ij}}(N)$ with $p_{ij}^{-1} = q_{ij} = \omega^{i-j}$.

The multiparameter quantum group is a Hopf algebra $\mathrm{GL}_{p_{ij},q_{ij}}(N)$ generated by N^2 elements x_{ij} and 1 with the following relations

$$
\begin{array}{ll}
x_{il}x_{ik} = q_{kl}x_{ik}x_{il} & k \leqslant l \\
x_{jk}x_{ik} = p_{ij}x_{ik}x_{jk} & i \leqslant j \\
x_{il}x_{jk} + q_{ij}x_{jl}x_{ik} - q_{kl}x_{ik}x_{jl} - q_{ij}q_{kl}x_{jk}x_{il} = 0 & i \leqslant j, k \leqslant l \\
x_{il}x_{jk} + p_{ij}^{-1}x_{jl}x_{ik} - p_{kl}^{-1}x_{ik}x_{jl} - p_{ij}^{-1}p_{kl}^{-1}x_{jk}x_{il} = 0 & i \leqslant j, k \leqslant l.
\end{array}
$$

The coproduct is defined as usual: $\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$. There is also a determinant element in $\mathrm{GL}_{p_{ij},q_{ij}}(N)$, which will not be needed explicitly. What we call the quantum group is actually the algebra obtained by adjoining the inverse of the determinant, with which the antipode is defined.

The quantum algebra $\mathrm{GL}_{p_{ij},q_{ij}}(N)$ with $p_{ij}^{-1} = q_{ij} = \omega^{i-j}$ has also the same representation in terms of the Weyl algebras. The assignment $\rho(x_{ij}) = \omega^{i-1}ZX^{j-i}$ gives the fundamental representation. Thus we have the following diagram

$$
\begin{tikzcd}
GL_{\omega^{i-j}}(N) \arrow[r, "?"] \arrow[d, "\rho"'] & A(S) \arrow[d, "\pi"] \\
End(\mathrm{C}^N) \arrow[r, equal] & End(\mathrm{C}^N)
\end{tikzcd}
$$

In the case of $N = 2$ the question mark arrow is a homomorphic mapping. We suspect that there is a possible relation between $\mathrm{GL}_{\omega^{i-j}}(N)$ and the algebra $A(S)$, which may require futher techniques to attack. If this is ture, it will be much easier to study the irreducible representations of $A(S)$.

We are sincerely grateful to Professor C N Yang for his interest in the work and encouragement. The exact formula of (21) was obtained in a discussion with him. We also thank H K Zhao and K Xue for discussions. Ge and Liu greatly acknowledge the support of the Chinese National Science Foundation.

References

[1] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
[2] Fateev V and Zamolodchikov A B 1982 *Phys. Lett.* A **92** 37–9
[3] Kashiwara M and Miwa T 1986 *Nucl. Phys.* B **275** 121–34
[4] Cherednik I V 1982 *Sov. J. Nucl. Phys.* **36(2)** 320–4
[5] Chudnovsky D V and Chudnovsky G V 1981 *Phys. Lett.* **81A** 105–10
[6] Tracy C 1985 *Physica* **16D** 203–20
[7] Ge M L, Zhao H K and Xue K 1990 *Phys. Lett.* **151A** 145–9
[8] Weyl H 1950 *The Theory of Groups and Quantum Mechanics* (New York: Dover)
[9] Faddeev L D, Reshetikhin N Y and Takhtajan L A 1990 *Leningrad Math. J.* **1** 193-225
[10] de Vega H J 1990 *Int. J. Mod. Phys.* B **4** 735
[11] Majid S 1990 *Int. J. Mod. Phys.* A **5** 1–91
[12] Belavin A A 1981 *Nucl. Phys.* B **180** 189–200
[13] Hasegawa K and Yamada Y 1990 *Phys. Lett.* **146A** 387–96
[14] Sklyanin E K 1982 *Funct. Anal.* **16** 263–9; 1983 *Funct. Anal.* **17** 273–84